ON THE CONTACT INTERACTION BETWEEN THE EDGES OF A CIRCULAR CRACK IN AN ELASTIC SEMI-SPACE AND AN ABSOLUTELY RIGID INCLUSION

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We assume that an absolutely rigid inclusion of a rather general configuration with a smooth surface is embedded into the central part of a circular crack in an elastic homogeneous isotropic space. It is supposed that the inclusion has central and axial symmetry and has the shape of an ellipsoid of revolution strongly flattened in the vertical direction.

It is also considered that upon indentation the inclusion adheres tightly to the crack edges along its entire length, and only normal contact stresses arise in the contact sections due to the smoothness of the inclusion surface. Under these assumptions, solving the described problem is reduced to solving the Fredholm IE of the first kind with a symmetric kernel, which is expressed by an incomplete elliptic function of the first kind.

By the collocation method in combination with the method of Gauss quadrature formulas at Chebyshev nodes is reduced to solving an SLAE of a rather simple structure. The main characteristics of fracture mechanics in this problem such as inclusion pressure on the crack edges, crack opening, normal breaking stresses outside the crack on its plane of location, SIF are represented by explicit formulas. In the particular case, when the radii of the inclusion and the crack are equal, an exact solution to the problem is obtained.

Here considered and their exact solutions are constructed by the method of factorization of triangular matrix-functions. But in the present paper, a simpler way to solve a governing IE suitable for engineering applications is proposed.

Problem formulation and derivation of basic equations

Let a homogeneous isotropic elastic space with elastic modulus E and Poisson's ratios ν, referred to the cylindrical coordinate system $r, \vartheta, z,$ contain a circular crack $\omega = \{z = 0;\, r \le a\}$ of radius a in the plane $z = 0$. Let further an absolutely rigid thin smooth inclusion be embedded into the central part $\omega_0 = \{z = 0;\, r \le b\}$ $(b < a)$ of the crack bounded by surfaces of revolution $z = \pm f(r)$, where $f(r)$ is a nonnegative function, continuous together with its first derivative in the interval $0 < r < b$ while $f(0) = \max\limits_{0 \le r \le b} f(r) = b$ $(b < a)$. The semi-minor axis d_1 of which is much smaller than the semi-major axis c_1, whence

$$f(r) = \frac{d_1}{c_1}\sqrt{c_1^2 - r^2} \quad (0 \le r \le b,\, b < c_1 < a). \qquad (1)$$

Because of the symmetry with respect to the plane $z=0$, the posed problem can be formulated as the following mixed boundary value problem of the theory of elasticity for the upper elastic half-space for translating the boundary conditions from the walls of the inclusion to the crack edges

$$\begin{cases} w(r,z)\big|_{z=+0} = f(r)(0<r\le b);\ w(r,z)\big|_{z=+0} = 0\ (r\ge a); \\ \tau_{rz}\big|_{z=+0} = 0\ (0\le r<\infty);\ \ \sigma_z(r,z)\big|_{z=+0} = 0\ (b<r<a); \\ \sigma_r(r,z), \sigma_z(r,z),\ \tau_{rz}(r,z)\to 0\ npu\ r^2+z^2\to\infty. \end{cases} \tag{2a-e}$$

Solving the problem (2a-e) we reduce to solving an IE. For this purpose, we first construct a solution to the auxiliary problem of the stress state of an elastic space with a circular crack ω; normal distributed forces of intensity $p(r)$ equal in magnitude and opposite in direction are applied to the upper and lower edges of the crack ($p(r)\equiv 0$ at $b<r<a$).

This auxiliary problem is equivalent to the following mixed boundary value problem for the upper elastic half-space:

$$\begin{cases} \sigma_z\big|_{z=+0} = -p(r) \ (0<r<a); \ \ w\big|_{z=+0} = 0 \ \ (r \geq a); \\ \sigma_z, \sigma_r, \tau_{rz} \to 0 \ npu \ \ r^2+z^2 \to \infty. \end{cases} \qquad \text{(3a-c)}$$

We have the following expression for the vertical displacements $w(r,+0)$ of the boundary points of the upper elastic half-space

$$w(r,+0) = \Theta \int_0^{\infty} \int_{-\pi}^{\pi} \frac{\sum(\rho)\rho d\rho d\varphi}{\sqrt{r^2+\rho^2-2r\rho\cos(\varphi-\vartheta)}} \qquad \left(\Theta = 1-\nu^2/\pi E; \quad 0<r<\infty\right). \qquad (4)$$

Therefore, formula (4) can be represented by the Weber-Sonin integral $R(r,\rho)$ in the form

$$w(r,+0) = 2\pi\Theta \int_0^{\infty} \sum(\rho)\rho d\rho \int_0^{\infty} J_0(\lambda r) J_0(\lambda\rho) d\lambda (0<r<\infty). \qquad (5)$$

After simple transformations we arrive at the following inversion formula for eq. (5) $(0<r<\infty)$:

$$\sum(r) = -\frac{1}{2\pi\Theta}\left(\frac{d}{dr}+\frac{1}{r}\right) \int_0^{a} \varphi'(\rho)\rho d\rho \int_0^{\infty} J_1(\lambda r) J_1(\lambda\rho) d\lambda \qquad (6)$$

Considering now the key equation (6) on the crack, we obtain the following integro-differential equation for half of the dislocation density on the crack edges $\varphi'(r)$, $(0<r<a)$

$$\left(\frac{d}{dr}+\frac{1}{r}\right)\int_0^a \varphi'(\rho)\rho d\rho\int_0^\infty J_1(\lambda r)J_1(\lambda\rho)d\lambda=-2\pi\Theta p(r) \qquad (7)$$

under the boundary condition $\varphi(a)=0$, which expresses the condition of the continuity of vertical displacements on the boundary circle of the crack $r=a$.

Since $V_1(0)=0$, then $C=0$ and, therefore, equation is equivalent to the integro-differential equation

$$\int_0^a \varphi'(\rho)\rho d\rho\int_0^\infty J_1(\lambda r)J_1(\lambda\rho)d\lambda=\frac{1}{r}\int_0^r ug(u)du \quad (0<r<a) \qquad (8)$$

under condition $\varphi(a)=0$.

Omitting intermediate transformations, we have

$$\varphi'(r)=w'(r,+0)=\frac{2}{\pi}\left[\frac{rG(a)}{\sqrt{a^2-r^2}}-r\int_r^a\frac{G'(t)dt}{\sqrt{t^2-r^2}}\right](0<r<a)$$
$$G(r)=-\frac{2\pi\Theta}{r}\int_0^r\frac{tp(t)dt}{\sqrt{t^2-r^2}},\quad G(a)=-\frac{2\pi\Theta}{a}\int_0^a\frac{tp(t)dt}{\sqrt{a^2-t^2}} \qquad \text{(9a-c)}$$

Taking into account the boundary condition $\varphi(a)=0$, after elementary transformations we obtain

$$w(r,+0)=4\Theta\int_0^a L(r,u)up(u)du \quad (0<r\le a)$$
$$L(r,u)=\int_{\max(r,u)}^{a}\frac{dt}{\sqrt{(t^2-r^2)(t^2-u^2)}}. \qquad \text{(10a-b)}$$

As a result, we arrive at the following governing Fredholm IE of the first kind with a symmetric kernel:

$$4\Theta\int_0^b L(r,\rho)\,p\,p(\rho)\,d\rho=f(r) \quad (0<r<b), \qquad (11)$$

whence the contact pressure of the inclusion on the crack edges $p(r)$ is determined.

In IE (11), we introduce the dimensionless quantities

$$x=r/b,\ s=\rho/b;\ p_0(x)=4(1-\nu^2)p(bx)/E,$$
$$f_0(x)=f(bx)/b;\ c=a/b>1. \qquad (12)$$

As a result, IE (11) is transformed into the following IE

$$\frac{1}{\pi}\int_0^1 L_0(x,s)sp_0(s)ds = f_0(x) \quad (0<x<1)$$
$$L_0(x,s) = \int_{\max(x,s)}^{c} \frac{du}{\sqrt{(u^2-r^2)(u^2-s^2)}}. \qquad (13a\text{-}b)$$

Then it is easy to obtain

$$L_0(x,s) = \begin{cases} \frac{1}{x} F\left(\arcsin\sqrt{\frac{c^2-x^2}{c^2-s^2}}, \frac{s}{x}\right) & (x>s); \\ \frac{1}{s} F\left(\arcsin\sqrt{\frac{c^2-s^2}{c^2-x^2}}, \frac{x}{s}\right) & (x<s); \end{cases} \qquad (14a\text{-}b)$$

where $F(\mu,t)$ is the incomplete elliptic integral of the first kind.

After passing to dimensionless quantities (12), we obtain

$$K_I^{(0)} = \frac{4(1-\nu^2)}{\sqrt{\pi b}E} \quad K_I = \frac{2}{\pi}\int_0^1 \frac{s\,p_0(s)ds}{\sqrt{c^2-s^2}}. \qquad (15)$$

By (10a-b), the crack opening $\Psi(r)$ outside the inclusion is calculated by the formula

$$\Psi(r) = 2w(r,+0) = 8\Theta\int_0^b L(r,u)u\,p(u)\,du \quad (b \le r \le a),$$

which in dimensionless quantities (12) takes the form

$$\Psi_0(x) = \frac{2}{\pi}\int_0^1 L_0(x,s)\,s\,p_0(s)\,ds \quad (1 \le x \le c)$$
$$\Psi_0(x) = \Psi(bx)/b. \qquad (16a\text{-}b)$$

After calculating the required integrals in dimensionless quantities, we obtain

$$\sigma_z^{(0)}(x,0) = \frac{K_I^{(0)}}{\sqrt{x^2-c^2}} + \frac{p_0(0)}{\pi}\left[\frac{\pi}{2} - 2\arcsin\left(\frac{c}{x}\right) + \arcsin\left(\frac{2c^2-x^2}{x^2}\right)\right] + \frac{2\sqrt{x^2-c^2}}{\pi}\int_0^1 \frac{s p_0(s)\,ds}{\left(s^2-x^2\right)\sqrt{c^2-x^2}} \quad (x > c,\ c = a/b > 1);$$
$$\sigma_z^{(0)}(x,0) = 4\left(1-\nu^2\right)\sigma_z(bx,0)\big/E. \qquad (17a\text{-}b)$$

In dimensionless form, we also write the function $f(r)$ from (1) which characterizes the inclusion surface in the particular case of a flattened ellipsoid of revolution, and the function $f(r)=\delta \ (0<r<b)$ in the case of a thin circular disk:

$$f_0(x)=\varepsilon\sqrt{1-\gamma x^2}; \quad f_0(x)=\delta_0 \quad \begin{pmatrix} 0<x<1; \ \varepsilon=d_1/b= \ 1; \\ \gamma=b/c_1; \ \delta_0=\gamma/b= \ 1 \end{pmatrix} \tag{18}$$

Thus, after the solution of the governing IE (13a-b) - (14a-b) has been constructed, the main characteristics of the considered problem in dimensionless form are determined by formulas (15), (16a-b) and (17a-b).

Reduction of the governing IE (13a-b)-(14a-b) to an SLAE

First, we extend equation (13a-b)-(14a-b) in an even way to an interval $(-1,0)$ and write it in the form

$$\frac{1}{2\pi}\int_{-1}^{1} M_0(x,s)|s|\varphi_0(s)\,ds = h_0(x) \quad (-1<x<1)$$

$$M_0(x,s) = L_0(|x|,|s|) = \begin{cases} \frac{1}{|x|}F\left(\arcsin\sqrt{\frac{c^2-x^2}{c^2-s^2}},\frac{|s|}{|x|}\right) & (|s|<|x|); \\ \frac{1}{|s|}F\left(\arcsin\sqrt{\frac{c^2-s^2}{c^2-x^2}},\frac{|x|}{|s|}\right) & (|x|<|s|); \end{cases} \qquad (19a\text{-}c)$$

$$\varphi_0(x) = p_0(|x|);\; h_0(x) = f_0(|x|) \; (-1<x<1).$$

As a result, we arrive at the SLAE

$$\sum_{m=1}^{N} M_{nm} X_m = a_n \quad \left(n = \overline{1,N}\right)$$

$$M_{nm} = \frac{|s_m|}{2N} \begin{cases} \frac{1}{|x_n|} F\left(\arcsin\sqrt{\frac{c^2 - x_n^2}{c^2 - s_m^2}}, \frac{|s_m|}{|x_n|}\right) & \left(|s_m| < |x_n|\right); \\ \frac{1}{|s_m|} F\left(\arcsin\sqrt{\frac{c^2 - s_m^2}{c^2 - x_n^2}}, \frac{|x_n|}{|s_m|}\right) & \left(|x_n| < |s_m|\right); \end{cases} \tag{20a-d}$$

$$X_m = \Omega_0\left(s_m\right); \; a_n = h_0\left(x_n\right) \qquad \left(m, n = \overline{1,N}\right).$$

Now we express all the characteristics of the problem under consideration in terms of the solution of the SLAE (20a-d).

Proceeding from formulas (15), (16a-b) и (17a-b), extended, as above, in a particular way on the interval $(-1,0)$, we have

$$K_I^{(0)} = \frac{1}{N} \sum_{m=1}^{N} \frac{|s_m| X_m}{\sqrt{c^2 - s_m^2}}; \tag{21}$$

$$\Psi_0(x) = \frac{1}{N}\sum_{m=1}^{N} M_0(x, s_m)|s_m| X_m \quad (1 \le |x| \le c); \tag{22}$$

$$\sigma_z^{(0)}(x,0) = \frac{K_I^{(0)}}{\sqrt{x^2 - c^2}} + \frac{p_0(0)}{\pi}\left[\frac{\pi}{2} - 2\arcsin\left(\frac{c}{x}\right) + \arcsin\left(\frac{2c^2 - x^2}{x^2}\right)\right] + \frac{\sqrt{x^2 - c^2}}{N}\sum_{m=1}^{N}\frac{|s_m| X_m}{(s_m^2 - x^2)\sqrt{c^2 - s_m^2}} \quad (x > c,\ c = a/b > 1). \tag{23}$$

Thus, the calculation formulas will be formulas (21)-(23).

Exact solution of the problem in a particular case

If it is granted that $b = a$, then in this particular case it is possible to construct an exact solution to the problem. Indeed, in this case $c = 1$, and the governing IE (13a-b) can be represented as

$$\frac{1}{\pi}\int_x^1 \frac{du}{\sqrt{u^2 - x^2}}\int_0^u \frac{s p_0(s)\,ds}{\sqrt{u^2 - s^2}} = f_0(x) \quad (0 < x < 1). \tag{24}$$

Assuming

$$X_0(u) = \int_0^u \frac{s p_0(s)\,ds}{\sqrt{u^2 - s^2}}, \tag{25}$$

$$\frac{1}{\pi}\int_x^1 \frac{X_0(u)\,du}{\sqrt{u^2-x^2}} = f_0(x) \quad (0<x<1). \tag{26}$$

Thus, in this particular case, solving the problem is reduced to the sequential solving two Abel integral equations (25) and (26). According to Abel's inversion formulas $(0<x<1)$

$$X_0(x) = -2\frac{d}{dx}\int_x^1 \frac{f_0(u)\,u\,du}{\sqrt{u^2-x^2}};\; p_0(x) = \frac{2}{\pi x}\frac{d}{dx}\int_0^x \frac{X_0(t)\,t\,dt}{\sqrt{x^2-t^2}}. \tag{27a-b}$$

Now let the inclusion have the shape of a flat circular disk. Then using the second formula (18) in (27a-b) we obtain

$$X_0(x) = -2\delta_0\frac{d}{dx}\int_x^1 \frac{u\,du}{\sqrt{u^2-x^2}} = \frac{2\delta_0 x}{\sqrt{1-x^2}} \quad (0<x<1)$$

$$p_0(x) = \frac{4\delta_0}{\pi x}\frac{d}{dx}\int_0^x \frac{t^2\,dt}{\sqrt{(1-t^2)(x^2-t^2)}} = \frac{4\delta_0}{\pi x}\frac{d}{dx}\left[x^2 D\left(\frac{\pi}{2},x\right)\right],$$

Using the formulas for the differentiation of the integrals $K(x)$ and $E(x)$ after simple transformations we obtain

$$p_0(x)=\frac{4\delta_0}{\pi}\frac{E(x)}{\left(1+\sqrt{1-x^2}\right)\sqrt{1-x^2}}\quad(0<x<1).$$

Then by (15)

$$\lim_{c\to 1}K_I^{(0)}=\frac{2\delta_0}{\pi^2}\lim_{c\to 1}\int_0^1\frac{sE(s)\,ds}{\sqrt{\left(1-s^2\right)\left(c^2-s^2\right)}\left(1+\sqrt{1-s^2}\right)}=\infty. \qquad (28)$$

On crack propagation

Relation (28) leads to the fact that in this simplest particular case, when passing to the limit , SIF infinitely increases, taking on its critical value for the given material, at which the crack begins to propagate. It turns out that as the crack propagates earlier than the end points of inclusion reach the crack tips, in full accordance with the principle of self-similarity of the end sections of the cracks. Consequently, the complete contact of the inclusion with the crack edges over its entire surface is physically unrealistic.

Based on the obtained analytical formulas for the characteristics of the problem under consideration, one can carry out their effective numerical analysis.

Conclusion

The results presented in the article can be used in studies of related problems when an absolute rigid thin inclusion contacts the edges of an interphase circular crack in a piecewise homogeneous space or in a piecewise homogeneous half-space consisting of a layer and a half-space. Further development of these results can also be used to study the propagation of an interfacial circular crack in contact with a rigid inclusion, to carry out a numerical analysis of the problem characteristics.


Thank you