ON THE CONTACT INTERACTION BETWEEN THE EDGES OF A CIRCULAR CRACK IN AN ELASTIC SEMI-SPACE AND AN ABSOLUTELY RIGID INCLUSION

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Introduction

The study on the interaction of stress concentrators such as cracks and inclusions with massive deformable bodies has both theoretical and practical significance. Local stress fields with large gradients are formed around these concentrators, often leading to crack propagation and significantly affecting the strength characteristics of structures and their parts. Such problems of stress concentration arise in the mechanics of composites, in geomechanics, in calculating the strength of various building structures, in thermoelasticity, in many areas of applied mechanics and in their engineering applications. Due to the relevance of these problems, they have become the subject of numerous studies. An especially large number of works have been published on the study of the interaction of circular cracks in deformable bodies with massive or thin inclusions. In this connection, we point out the works [1-4]. Numerous results on this topic are summarized in the SIF handbooks [5,6].

The present paper generalizes the problem. Precisely, we assume that an absolutely rigid inclusion of a rather general configuration with a smooth surface is embedded into the central part of a circular crack in an elastic homogeneous isotropic space. It is supposed that the inclusion has central and axial symmetry and has the shape of an ellipsoid of revolution strongly flattened in the vertical direction. The inclusion in particular can have the form of a thin circular disk. It is also considered that upon indentation the inclusion adheres tightly to the crack edges along its entire length, and only normal contact stresses arise in the contact sections due to the smoothness of the inclusion surface. Under these assumptions, solving the described problem is reduced to solving the Fredholm IE of the first kind with a symmetric kernel, which is expressed by an incomplete elliptic function of the first kind. This equation, in turn, by the collocation method in combination with the method of Gauss quadrature formulas at Chebyshev nodes is reduced to solving an SLAE of a rather simple structure. The main characteristics of fracture mechanics in this problem such as inclusion pressure on the crack edges, crack opening, normal breaking stresses outside the crack on its plane of location, SIF are represented by explicit formulas. A brief discussion of crack propagation is carried out. In the particular case, when the radii of the inclusion and the crack are equal, an exact solution to the problem is obtained.

Here considered and their exact solutions are constructed by the method of factorization of triangular matrix-functions. But in the present paper, a simpler way to solve a governing IE suitable for engineering applications is proposed.

Problem formulation and derivation of basic equations

Let a homogeneous isotropic elastic space with elastic modulus E and Poisson's ratios ν, referred to the cylindrical coordinate system $r, \vartheta, z,$ contain a circular crack $\omega = \{z = 0;\, r \le a\}$ of radius a in the plane $z = 0$. Let further an absolutely rigid thin smooth inclusion be embedded into the central part $\omega_0 = \{z = 0;\, r \le b\}$ $(b < a)$ of the crack bounded by surfaces of revolution $z = \pm f(r)$, where $f(r)$ is a nonnegative function, continuous together with its first derivative in the interval $0 < r < b$ while $f(0) = \max\limits_{0 \le r \le b} f(r) =$ b $(b < a)$. The semi-minor axis d_1 of which is much smaller than the semi-major axis c_1, whence

$$f(r) = \frac{d_1}{c_1}\sqrt{c_1^2 - r^2} \quad (0 \le r \le b,\, b < c_1 < a). \tag{1}$$

Further, we assume that a thin inclusion after indentation into the crack edges adheres tightly to them along the contact section in the shape of a circle $r \le b$ and causes an axisymmetric deformation of the elastic space. Due to the smoothness of the inclusion surface, we will suppose that only normal stresses arise on the contact circle.

Under the assumptions made, it is required to determine the above-mentioned characteristics of the problem under consideration.

Because of the symmetry with respect to the plane $z = 0$, the posed problem can be formulated as the following mixed boundary value problem of the theory of elasticity for the upper elastic half-space for translating the boundary conditions from the walls of the inclusion to the crack edges

$$\begin{cases} w(r,z)\big|_{z=+0} = f(r)(0 < r \le b);\; w(r,z)\big|_{z=+0} = 0\;(r \ge a); \\ \tau_{rz}\big|_{z=+0} = 0\;\;(0 \le r < \infty);\;\; \sigma_z(r,z)\big|_{z=+0} = 0\;\;(b < r < a); \\ \sigma_r(r,z), \sigma_z(r,z),\; \tau_{rz}(r,z) \to 0 \;\textit{npu}\;\; r^2 + z^2 \to \infty. \end{cases} \tag{2a-e}$$

Here $w(r,z)$ is the vertical displacement (in the direction of the z-axis) of the upper elastic half-space point $M(r, \vartheta, z)$, $\sigma_r, \sigma_z, \tau_{rz}$ are the components of normal and tangential stresses respectively. Solving the problem (2a-e) we reduce to solving an IE. For this purpose, we first construct a solution to the auxiliary problem of the stress state of an elastic space with a circular crack ω; normal distributed forces of intensity $p(r)$ equal in magnitude and opposite in direction are applied to the upper and lower edges of the crack ($p(r) \equiv 0$ at $b < r < a$). Again, due to the symmetry with respect to the plane $z = 0$, this auxiliary problem is equivalent to the following mixed boundary value problem for the upper elastic half-space:

$$\begin{cases} \sigma_z\big|_{z=+0} = -p(r)\;\;(0 < r < a);\;\; w\big|_{z=+0} = 0\;\;(r \ge a); \\ \sigma_z, \sigma_r,\, \tau_{rz} \to 0 \;\textit{npu}\;\; r^2 + z^2 \to \infty. \end{cases} \tag{3a-c}$$

The solution of the problem (3a-c) in displacements can be immediately obtained using the well-known Boussinesq problem for an elastic half-space in combination with the linear principle of superposition of displacements. Assuming

$$\sigma_z\big|_{z=+0} = -\sum(r) = \begin{cases} -p(r) & (0<r<a); \\ -\sigma(r) & (r>a) \end{cases}$$

we have the following expression for the vertical displacements $w(r,+0)$ of the boundary points of the upper elastic half-space

$$w(r,+0) = \Theta\int_0^\infty\int_{-\pi}^{\pi} \frac{\sum(\rho)\rho d\rho d\varphi}{\sqrt{r^2+\rho^2-2r\rho\cos(\varphi-\vartheta)}} \qquad \left(\Theta = 1-\nu^2/\pi E; \quad 0<r<\infty\right). \tag{4}$$

Therefore, formula (4) can be represented by the Weber-Sonin integral $R(r,\rho)$ in the form

$$w(r,+0) = 2\pi\Theta\int_0^\infty \sum(\rho)\rho d\rho \int_0^\infty J_0(\lambda r)J_0(\lambda\rho)d\lambda \, (0<r<\infty). \tag{5}$$

Now, considering equality (5) as an IE with respect to $\sum(r)$, we invert it, for which we change the order of integration on the right-hand side and write this formula as

$$\int_0^\infty J_0(\lambda r)\bar{\Sigma}(\lambda)d\lambda = \frac{\varphi(r)}{2\pi\Theta} \quad (0<r<\infty).$$

$$\varphi(r) = w(r,+0), \; \bar{\Sigma}(\lambda) = \int_0^\infty \sum(\rho)\rho J_0(\lambda\rho)d\rho.$$

After simple transformations we arrive at the following inversion formula for eq. (5) $(0<r<\infty)$:

$$\sum(r) = -\frac{1}{2\pi\Theta}\left(\frac{d}{dr}+\frac{1}{r}\right)\int_0^a \varphi'(\rho)\rho d\rho\int_0^\infty J_1(\lambda r)J_1(\lambda\rho)d\lambda \tag{6}$$

where $J_1(x)$ is the Bessel function of the first kind with index 1. Equation (6) will be called the key equation of the problem under discussion.

Considering now the key equation (6) on the crack, we obtain the following integro-differential equation for half of the dislocation density on the crack edges $\varphi'(r)$, $(0<r<a)$

$$\left(\frac{d}{dr}+\frac{1}{r}\right)\int_0^a \varphi'(\rho)\rho d\rho\int_0^\infty J_1(\lambda r)J_1(\lambda\rho)d\lambda = -2\pi\Theta p(r) \tag{7}$$

under the boundary condition $\varphi(a)=0$, which expresses the condition of the continuity of vertical displacements on the boundary circle of the crack $r=a$.

Assuming

$$V_1(r) = \int_0^a \varphi'(\rho)\rho d\rho\int_0^\infty J_1(\lambda r)J_1(\lambda\rho)d\lambda,$$

Since $V_1(0)=0,$ then $C=0$ and, therefore, equation is equivalent to the integro-differential equation

$$\int_0^a \varphi'(\rho)\rho d\rho\int_0^\infty J_1(\lambda r)J_1(\lambda\rho)d\lambda = \frac{1}{r}\int_0^r ug(u)du \quad (0<r<a) \tag{8}$$

under condition $\varphi(a)=0$.

Omitting intermediate transformations, we have

$$\begin{gathered}\varphi'(r) = w'(r,+0) = \frac{2}{\pi}\left[\frac{rG(a)}{\sqrt{a^2-r^2}} - r\int_r^a \frac{G'(t)\,dt}{\sqrt{t^2-r^2}}\right](0<r<a)\\ G(r) = -\frac{2\pi\Theta}{r}\int_0^r \frac{tp(t)\,dt}{\sqrt{t^2-r^2}},\ G(a) = -\frac{2\pi\Theta}{a}\int_0^a \frac{tp(t)\,dt}{\sqrt{a^2-t^2}}\end{gathered} \qquad \text{(9a-c)}$$

Taking into account the boundary condition $\varphi(a)=0$, after elementary transformations we obtain

$$\begin{gathered}w(r,+0) = 4\Theta\int_0^a L(r,u)up(u)\,du \quad (0<r\le a)\\ L(r,u) = \int_{\max(r,u)}^a \frac{dt}{\sqrt{(t^2-r^2)(t^2-u^2)}}.\end{gathered} \qquad \text{(10a-b)}$$

As a result, we arrive at the following governing Fredholm IE of the first kind with a symmetric kernel:

$$4\Theta\int_0^b L(r,\rho)\,p\,p(\rho)\,d\rho = f(r) \quad (0<r<b), \qquad (11)$$

whence the contact pressure of the inclusion on the crack edges $p(r)$ is determined.

In IE (11), we introduce the dimensionless quantities

$$\begin{gathered}x = r/b,\ s = \rho/b;\ p_0(x) = 4(1-\nu^2)\,p(bx)/E,\\ f_0(x) = f(bx)/b;\ c = a/b > 1.\end{gathered} \qquad (12)$$

As a result, IE (11) is transformed into the following IE

$$\begin{gathered}\frac{1}{\pi}\int_0^1 L_0(x,s)sp_0(s)\,ds = f_0(x) \quad (0<x<1)\\ L_0(x,s) = \int_{\max(x,s)}^c \frac{du}{\sqrt{(u^2-r^2)(u^2-s^2)}}.\end{gathered} \qquad \text{(13a-b)}$$

Then it is easy to obtain

$$L_0(x,s) = \begin{cases}\dfrac{1}{x}F\left(\arcsin\sqrt{\dfrac{c^2-x^2}{c^2-s^2}},\dfrac{s}{x}\right) & (x>s);\\ \dfrac{1}{s}F\left(\arcsin\sqrt{\dfrac{c^2-s^2}{c^2-x^2}},\dfrac{x}{s}\right) & (x<s);\end{cases} \qquad \text{(14a-b)}$$

where $F(\mu,t)$ is the incomplete elliptic integral of the first kind.

After passing to dimensionless quantities (12), we obtain

$$K_I^{(0)} = \frac{4(1-\nu^2)}{\sqrt{\pi b}E} \quad K_I = \frac{2}{\pi}\int_0^1 \frac{s\,p_0(s)\,ds}{\sqrt{c^2-s^2}}. \qquad (15)$$

By (10a-b), the crack opening $\Psi(r)$ outside the inclusion is calculated by the formula

$$\Psi(r) = 2w(r,+0) = 8\Theta\int_0^b L(r,u)u\,p(u)\,du \quad (b\le r\le a),$$

which in dimensionless quantities (12) takes the form

$$\begin{gathered}\Psi_0(x) = \frac{2}{\pi}\int_0^1 L_0(x,s)\,s\,p_0(s)\,ds \quad (1\le x\le c)\\ \Psi_0(x) = \Psi(bx)/b.\end{gathered} \qquad \text{(16a-b)}$$

To determine the breaking normal stresses outside the crack $\sigma_z(r,0)$ in the plane $z=0$, we again turn to the key equation (6) and consider it outside the crack, i.e., at $r>a$:

$$\sigma_z(r,0)=\frac{1}{2\pi\Theta}\left(\frac{d}{dr}+\frac{1}{r}\right)\int_0^a \varphi'(\rho)\rho d\rho\int_0^\infty J_1(\lambda r)J_1(\lambda\rho)d\lambda.$$

Next, we substitute here the expression of $\varphi'(r)$ from (9a-c). After calculating the required integrals in dimensionless quantities, we obtain

$$\sigma_z^{(0)}(x,0)=\frac{K_I^{(0)}}{\sqrt{x^2-c^2}}+\frac{p_0(0)}{\pi}\left[\frac{\pi}{2}-2\arcsin\left(\frac{c}{x}\right)+\arcsin\left(\frac{2c^2-x^2}{x^2}\right)\right]+\frac{2\sqrt{x^2-c^2}}{\pi}\int_0^1\frac{sp_0(s)ds}{\left(s^2-x^2\right)\sqrt{c^2-x^2}}\quad (x>c,\ c=a/b>1);\tag{17a-b}$$
$$\sigma_z^{(0)}(x,0)=4\left(1-\nu^2\right)\sigma_z(bx,0)\big/E.$$

In dimensionless form, we also write the function $f(r)$ from (1) which characterizes the inclusion surface in the particular case of a flattened ellipsoid of revolution, and the function $f(r)=\delta\ (0<r<b)$ in the case of a thin circular disk:

$$f_0(x)=\varepsilon\sqrt{1-\gamma x^2};\quad f_0(x)=\delta_0\ \begin{pmatrix}0<x<1;\ \varepsilon=d_1/b=\ 1;\\ \gamma=b/c_1;\ \delta_0=\gamma/b=\ 1\end{pmatrix}\tag{18}$$

Thus, after the solution of the governing IE (13a-b) - (14a-b) has been constructed, the main characteristics of the considered problem in dimensionless form are determined by formulas (15), (16a-b) and (17a-b).

Reduction of the governing IE (13a-b)-(14a-b) to an SLAE

First, we extend equation (13a-b)-(14a-b) in an even way to an interval $(-1,0)$ and write it in the form

$$\frac{1}{2\pi}\int_{-1}^{1}M_0(x,s)|s|\varphi_0(s)ds=h_0(x)\quad(-1<x<1)$$

$$M_0(x,s)=L_0(|x|,|s|)=\begin{cases}\dfrac{1}{|x|}F\left(\arcsin\sqrt{\dfrac{c^2-x^2}{c^2-s^2}},\dfrac{|s|}{|x|}\right)\ (|s|<|x|);\\[2ex] \dfrac{1}{|s|}F\left(\arcsin\sqrt{\dfrac{c^2-s^2}{c^2-x^2}},\dfrac{|x|}{|s|}\right)\ (|x|<|s|);\end{cases}\tag{19a-c}$$
$$\varphi_0(x)=p_0(|x|);\ h_0(x)=f_0(|x|)\ (-1<x<1).$$

Then, we apply the collocation method to equation (19a-c) and use the Gauss quadrature formulas at Chebyshev's nodes to calculate the required integrals. For this, we represent the solution of equation (19a-c) in the form

$$\varphi_0(x)=\Omega_0(x)\big/\sqrt{1-x^2}\ \ (-1<x<1),$$

where $\Omega_0(x)$ is the Hölder function on the interval $[-1,1]$. As internal nodes we take points

$$s_m=\cos\left[(2m-1)\pi/2N\right]\ \left(m=\overline{1,N}\right),$$

i.e. roots of the equation $T_N(s)=0$, where $T_N(s)$ are the Chebyshev polynomials of the first kind, while as the outer nodes, we take points

$$x_n=\cos\left[\pi n/(N+1)\right]\ \left(n=\overline{1,N}\right),$$

i.e. roots of the equation $U_N(x)=0$, where $U_N(x)$ are the Chebyshev polynomials of the second kind; N is any natural number. As a result, we arrive at the SLAE

$$\sum_{m=1}^{N} M_{nm} X_m = a_n \quad \left(n=\overline{1,N}\right)$$
$$M_{nm} = \frac{|s_m|}{2N} \begin{cases} \dfrac{1}{|x_n|} F\left(\arcsin\sqrt{\dfrac{c^2-x_n^2}{c^2-s_m^2}}, \dfrac{|s_m|}{|x_n|}\right) & \left(|s_m|<|x_n|\right); \\ \dfrac{1}{|s_m|} F\left(\arcsin\sqrt{\dfrac{c^2-s_m^2}{c^2-x_n^2}}, \dfrac{|x_n|}{|s_m|}\right) & \left(|x_n|<|s_m|\right); \end{cases} \tag{20a-d}$$
$$X_m = \Omega_0(s_m);\; a_n = h_0(x_n) \qquad \left(m,n=\overline{1,N}\right).$$

Now we express all the characteristics of the problem under consideration in terms of the solution of the SLAE (20a-d).

Proceeding from formulas (15), (16a-b) и (17a-b), extended, as above, in a particular way on the interval $(-1,0)$, we have

$$K_I^{(0)} = \frac{1}{N}\sum_{m=1}^{N} \frac{|s_m| X_m}{\sqrt{c^2-s_m^2}}; \tag{21}$$

$$\Psi_0(x) = \frac{1}{N}\sum_{m=1}^{N} M_0(x,s_m)|s_m| X_m \quad \left(1\le |x| \le c\right); \tag{22}$$

$$\sigma_z^{(0)}(x,0) = \frac{K_I^{(0)}}{\sqrt{x^2-c^2}} + \frac{p_0(0)}{\pi}\left[\frac{\pi}{2} - 2\arcsin\left(\frac{c}{x}\right) + \arcsin\left(\frac{2c^2-x^2}{x^2}\right)\right] + \frac{\sqrt{x^2-c^2}}{N}\sum_{m=1}^{N} \frac{|s_m| X_m}{\left(s_m^2-x^2\right)\sqrt{c^2-s_m^2}} \quad \left(x>c,\; c=a/b>1\right). \tag{23}$$

Thus, the calculation formulas will be formulas (21)-(23).

Exact solution of the problem in a particular case

If it is granted that $b=a$, then in this particular case it is possible to construct an exact solution to the problem. Indeed, in this case $c=1$, and the governing IE (13a-b) can be represented as

$$\frac{1}{\pi}\int_x^1 \frac{du}{\sqrt{u^2-x^2}} \int_0^u \frac{s p_0(s)\,ds}{\sqrt{u^2-s^2}} = f_0(x) \quad \left(0<x<1\right). \tag{24}$$

Assuming

$$X_0(u) = \int_0^u \frac{s p_0(s)\,ds}{\sqrt{u^2-s^2}}, \tag{25}$$

we write (24) as

$$\frac{1}{\pi}\int_x^1 \frac{X_0(u)\,du}{\sqrt{u^2-x^2}} = f_0(x) \quad \left(0<x<1\right). \tag{26}$$

Thus, in this particular case, solving the problem is reduced to the sequential solving two Abel integral equations (25) and (26). According to Abel's inversion formulas $\left(0<x<1\right)$

$$X_0(x) = -2\frac{d}{dx}\int_x^1 \frac{f_0(u)\,u\,du}{\sqrt{u^2-x^2}};\; p_0(x) = \frac{2}{\pi x}\frac{d}{dx}\int_0^x \frac{X_0(t)\,t\,dt}{\sqrt{x^2-t^2}}\;. \tag{27a-b}$$

Now let the inclusion have the shape of a flat circular disk. Then using the second formula (18) in (27a-b) we obtain

$$X_0(x) = -2\delta_0 \frac{d}{dx}\int_x^1 \frac{udu}{\sqrt{u^2-x^2}} = \frac{2\delta_0 x}{\sqrt{1-x^2}} \quad (0<x<1)$$
$$p_0(x) = \frac{4\delta_0}{\pi x}\frac{d}{dx}\int_0^x \frac{t^2dt}{\sqrt{(1-t^2)(x^2-t^2)}} = \frac{4\delta_0}{\pi x}\frac{d}{dx}\left[x^2 D\left(\frac{\pi}{2},x\right)\right],$$

where $D\left(\frac{\pi}{2},x\right)$ is the full elliptical integral of the third kind.

$$D\left(\frac{\pi}{2},x\right) = \left[K(x)-E(x)\right]/x^2 \quad (0<x<1),$$

where $E(x)$ is the full elliptical integral of the second kind. Using the formulas for the differentiation of the integrals $K(x)$ and $E(x)$ after simple transformations we obtain

$$p_0(x) = \frac{4\delta_0}{\pi}\frac{E(x)}{\left(1+\sqrt{1-x^2}\right)\sqrt{1-x^2}} \quad (0<x<1).$$

Then by (15)

$$\lim_{c\to 1} K_I^{(0)} = \frac{2\delta_0}{\pi^2}\lim_{c\to 1}\int_0^1 \frac{sE(s)ds}{\sqrt{(1-s^2)(c^2-s^2)}\left(1+\sqrt{1-s^2}\right)} = \infty. \qquad (28)$$

On crack propagation

Relation (28) leads to the fact that in this simplest particular case, when passing to the limit $c \to 1$, SIF, $K_I^{(0)}$, infinitely increases, taking on its critical value for the given material, at which the crack begins to propagate. It turns out that as $c \to 1$ the crack propagates earlier than the end points of inclusion reach the crack tips, in full accordance with the principle of self-similarity of the end sections of the cracks [17]. Consequently, the complete contact of the inclusion with the crack edges over its entire surface is physically unrealistic, although a passage to the limit in the solution to the original problem can formally be done.

Based on the obtained analytical formulas for the characteristics of the problem under consideration, one can carry out their effective numerical analysis.

Conclusion

The results presented in the article can be used in studies of related problems when an absolute rigid thin inclusion contacts the edges of an interphase circular crack in a piecewise homogeneous space or in a piecewise homogeneous half-space consisting of a layer and a half-space. Further development of these results can also be used to study the propagation of an interfacial circular crack in contact with a rigid inclusion, to carry out a numerical analysis of the problem characteristics.

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